Q4, August 2016

Prove that the determinant of an n x n matrix is (-1)^n times the constant term of the characteristic polynomial of A.

$$\chi_A(x) = \det(xI - A)$$

To obtain the constant term of the characteristic polynomial, plug in x = 0.

$$\chi_A(0) = \det(-A) = (-1)^n \det(A)$$

But the constant term of the characteristic polynomial is det(A).

Q4, August 2015

Let A be an n x n matrix. Let T be the linear operator that is multiplication by A. Prove that the minimal polynomial of T is the minimal polynomial of A.

$$f(T) = 0 \iff f(T)B = 0 \text{ for all } B.$$

$$\iff f(A)B = 0 \text{ for all } B. \iff f(A) = 0$$

The forward arrows show that the minimal polynomial of A divides the minimal polynomial of T. This is because any polynomial that annihilates T is divisible by the minimal polynomial of T. In particular, the minimal polynomial of T annihilates T, but by the forward arrows, it must annihilate A, hence it must be divisible by the minimal polynomial of A.

The reverse arrows show that the minimal polynomial of T divides the minimal polynomial of A. Because they are monic polynomials that divide each other, they are equal.

Q6(a, b), January 2015

Let A be an n x n matrix. Let T be the linear operator on IR^n defined by T(v) = Av. Consider the set W = {v \in IR^n | T(v) = v}. Suppose that nullity(T) + dim(W) = n.

(a.) Find the minimal polynomial of A.

Question: How can we find the minimal polynomial of A if we know nothing about A?

Observation: (Raneeta) We have that W = ker(T - I).

Observation: (Raneeta) The minimal polynomial of T is equal to the minimal polynomial of A. (We just showed that in a previous question.) This is because T is the matrix of A with respect to the standard basis of IR^n, i.e., e_i is the column with 1 in ith column and 0s elsewhere.

By Raneeta's first observation, we have that dim ker(T) + dim ker(T - I) = n.

If their sum is direct (i.e., their intersection is trivial), then ker(T) + ker(T - I) = IR^n. But this is true because if v is in both ker(T) and ker(T - I), then Av = 0 and Av = v, hence v = 0.

Because IR^n can be uniquely decomposed as ker(T) + ker(T - I), every real column vector v can be written uniquely as v = u + w, where u belongs to ker(T) and w belongs to ker(T - I).

  (T - I)(v) = (T - I)(u + w) = (T - I)(u) + (T - I)(w) = -u
T(T - I)(v) = T(-u) = 0

Observe that T(T - I) is the zero operator, hence its minimal polynomial divides x(x - 1).

1.) If W is zero, then T is zero, hence its minimal polynomial is x and JCF(T) = 0.
2.) If W is IR^n, then T = I, and its minimal polynomial is x - 1 and JCF(T) = I.
3.) Otherwise, the minimal polynomial of T is x(x - 1) by what we just said.

b.) Suppose that the minimal polynomial is x(x - 1).

1.) The invariant factors are n - 2 copies of x and x(x - 1). The elementary divisors are n - 1 copies of x and one copy of x - 1. So, JCF(T) = diag{0, 0, …, 0, 1} with n - 1 0s.

2.) The invariant factors are n - 2 copies of x - 1 and x(x - 1). The elementary divisors are n - 1 copies of x - 1 and one copy of x. So, JCF(T) = diag{1, 1, …, 1, 0} with n - 1 1s.

3.) The invariant factors are i copies of x and j > 1 copies of x(x - 1). The elementary divisors are i + j copies of x and j copies of x - 1. So, JCF(T) = diag{0, 0, …, 1, 1, …, 1} with i + j 0s and j 1s. Note that we could say the same with x - 1 in place of x.

Big Brain Observation: If the minimal polynomial is x(x - 1), then the elementary divisors are i copies of x and j copies of x - 1, so the Jordan Canonical Form is diag{0, 0, …., 0, 1, 1, …, 1} with i copies of 0 and j copies of 1 appearing on the diagonal.

Q1, August 2021

Let G be the abelian group Z x Z. Let N = $\langle(4, 1), (6, 3)\rangle$. Find an explicit isomorphism from G/N to a direct product of cylic groups.

This is equivalent to finding the Smith Normal Form of the matrix A whose rows are (4, 1) and (6, 3). We accomplish this using a sequence of elementary row and column operations on A.

$$A = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix} \underset{C_1 \leftrightarrow C_2}{\sim} \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix} \underset{C_2 - 4C_1 \mapsto C_2}{\sim} \begin{pmatrix} 1 & 0 \\ 3 & -6 \end{pmatrix} \underset{R_2 - 3R_1 \mapsto R_2}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix}$$

$$\underset{-C_2 \mapsto C_2}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

$$SNF(A) = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\frac{G}{N} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \times \frac{\mathbb{Z}}{6\mathbb{Z}} \cong \mathbb{Z}/6\mathbb{Z}$$